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ABSTRACT

A test was designed to measure whether or not the child's conceptualization of space would demand that two topological properties of figures (connectedness, and openness and closedness) remain invariant when the figures were subjected to transformations. Ninety children, 30 from each of ages four, six, and eight, were tested. Six items were given to each child in four modes; for each item the child indicated a "most like" and a "worst" preference. Results showed there was no significant interaction between age and mode, that age had a significant effect on the "worst" preference scores but not on "most like" scores, that test mode affected both preference scores, and that for two of the test modes the mean scores based on number of times copies not topologically equivalent to the model were selected as being "most like" the model were significantly above chance means of all age levels. The conclusion saw that the results do not support the theory that topological concepts develop prior to Euclidean and projective concepts in the child's representational space. (DT)

AN INVESTIGATION OF THE DEVELOPMENT OF SELECTED TOPOLOGICAL PROPERTIES IN THE REPRESENTATIONAL SPACE OF YOUNG CHILDREN

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A task of mathematics educators is to introduce various mathematical concepts into a comprehensive mathematics program for kindergarten through grade 12. The sequencing of these concepts is sometimes determined by the mathematics involved, sometimes by the experiences deemed appropriate for the student's needs in his physical surroundings, sometimes by the cognitive readiness of the student, sometimes by tradition and sometimes by mere chance. In the case of geometry, the sequence has traditionally followed the order of the historical development of geometry, that is, from Euclidean geometry to the more general geometries.

Jean Piaget [14] contends that the child's conception of space actually develops in reverse order to this traditional approach. He asserts that the Euclidean concepts develop first in the child, deriving from the topological concepts, then, are the projective and Euclidean concepts.

The major purpose of this study was to test Piaget's hypothesis, as outlined in The Child's Conception of Space, that the four-year-old child's representational space is predominantly topological in nature.

ORIENTATION TO THE PROBLEM

An analysis of Piaget's experiments and those of his replicators concerning the child's development of

topological concepts is the subject of another paper. However, some mathematical and psychological background will prove useful for purposes of this paper.

Mathematical Background

Ordinarily, a person regards a transformation as a change. In mathematics a transformation may be regarded as a rule associating points of a set X with points of another set Y. The transformation of special interest in topology is the homeomorphism. If a homeomorphism exists between the sets X and Y, then we say X and Y are homeomorphic or topologically equivalent. A popular misconception about homeomorphisms is that if two sets are homeomorphic (topologically equivalent), then it is possible to deform one into the other by pulling, bending, stretching, and shrinking as long as we do not tear or break. This is in error. However, the converse is true. That is, a transformation which involves no more than pulling, bending, stretching, shrinking, and in general, shape distortion is a homeomorphism. Thus if set Y can be obtained by performing such distortions on a set X, X and Y are topologically equivalent sets. For purposes of this paper, this intuitive notion of homeomorphism and topological equivalence will suffice. More rigorous statements are available in all standard topology textbooks and there is a rather discursive treatment in the August-September, 1960, edition of The American

Mathematical Monthly [1]. A property of a set X is said to be a topological property if any set homeomorphic to set X possesses that same property. Thus topological properties are those properties of a set which remain invariant under homeomorphisms.

Some examples of topological properties are interior of a set, exterior of a set, boundary of a set, connectedness ("one-piece-ness"), and openness and closedness of curves. Circles, triangles, and rectangles are all topologically equivalent. Some examples of non-topological properties are straightness, length of lines, and size of angles.

Contrasted with the topological transformation (homeomorphism) is the Euclidean transformation. Roughly speaking, a Euclidean transformation preserves distances between pairs of points. As a result it will also preserve shape and size. These transformations are also called "rigid motions." A Euclidean property of a set X is a property preserved by a Euclidean transformation.

The Euclidean properties will include all the above listed topological properties as well as those non-topological properties cited. The topological properties of a figure form a subset of the Euclidean properties of that figure. Note that the properties of a figure cannot be partitioned into Euclidean and topological properties. These categories are not disjoint. It would be meaningless to state that a given figure is a topological figure or a Euclidean figure.

Psychological Background

Piaget and Inhelder [14] have described a comprehensive theory of the child's conception of space. They suggest that the child's first concepts of space are topological ones. The theory is consistent with Piaget's developmental theory and in particular with Piaget, Inhelder, and Szeminska's [15] findings concerning the child's concept of measurement. In this work one finds

> The transition from topological relations to Euclidean relations cannot be effected without the elaboration of the twin concepts of distance and change of position. These developments are simultaneous in their occurence [15, p. 392].

Piaget's notion of concept is somewhat different from that held by many psychologists according to Elkind [6]. A brief comparison between the Piagetian and the discriminative response versions with respect to environmental variability and content follows.

Environmental variability can be regarded as being of two major types, variability between things and variability within things [6]. The first type confronts one with problems of similarity and difference, but the similarities and differences are those of spatially discrete objects. The second type deals with variability due to changes of state within an object or due to transformation on an object. Consideration of the first type of variability will determine whether or not an object will remain in that class after some change. For example

a vehicle may be classified as an automobile because it has selected similarities with other automobiles and differences from non-automobiles. But that automobile may be painted, wrecked, or have a flat tire. Is it still an automobile to the child? This question would be answered on the basis of the within variability of the child's concept. According to Elkind [6], Piaget is concerned with both types of variability in his conservation tasks.

Just as there are two types of variability, every concept has two different kinds of content. The discriminative [illegible] version of the concept concerns itself with the realm of objects that the concept points to or denotes. These exemplars of the concept constitute what is called the extensive content. But Piaget's conservation problems are concerned with the assessment of intensive content as well as extensive content. The intensive content corresponds to the common features connoted by all the exemplars of a concept. To Piaget, the "intensive content of a concept is always relative to the transformations that leave it invariant [6, p. 181]." As Elkind says, "When the intensive content is conceived in terms of transformations that leave the exemplar a member of the class, we have a direct and novel way of determining which properties are regarded critical to the concept [6, p. 182]."

Piaget's emphasis on invariability through transformations is especially interesting in view of the definition of a geometry presented by the famous mathematician Felix Klein in 1872. "A geometry is the study of those properties of a set X which remain invariant when the elements of X are subjected to the transformations of some transformation group [18, p. 70]." The notions, Piaget's and Klein's, dovetail nicely and have appeal as a means of studying the child's concept of space.

In The Child's Conception of Space, Piaget is investigating the child's representational space. This is contrasted with perceptual space.

> Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived [14, p. 17].

To illustrate, very young babies can distinguish between circular shapes and triangular shapes, but it is not until much later that they can represent these figures to themselves in thought [10, p.].

Representational space grows as a result of organization of actions performed on objects in space. At first these actions are sensorimotor in nature. They progress through an increasingly complex coordination of overt actions and displacements until they reach the

state of internalized actions. The appearance of the symbolic function, around the age of two years, enables the child to act not only on objects which are real and physically present but also on objects whose presence is only imagined. However, mental representation is not merely a recall from a memory bank. It is an active reconstruction of an object at the symbolic level.

Based on this theory, the representational space of an adult would be quite different from that of a child. That of the adult would be the result of years of active manipulations of his spatial environment. The child has not yet experienced such active manipulation. Thus representational space is the result of a developing process and not the result of an immediate perceptual "mental snapshot" of the environment.

To summarize, representational space in the child develops slowly, beginning with the advent of the symbolic function at about age two years. The symbolic function allows crude internalized actions. But these early internalizations cannot be coordinated into a system of actions. Gradually they become more complex, become reversible in thought, and can be combined into systems of actions. Piaget, in The Child's Conception of Space, traces this development process through successive steps. He asserts that the child's representational space is predominantly topological in nature until about six years of age. The representational space begins to expand

to include projective and Euclidean concepts at about six or seven years of age.

THE CURRENT STUDY

As was discussed above, according to Elkind an important aspect of concept attainment to Piaget is invariance under transformation. A test was designed intended to measure whether or not the child's conceptualization of space would demand that topological properties of figures remain invariant when those figures were subjected to transformations. The particular topological properties selected for study were connectedness and openness and closedness of curves. Connectedness was selected for study because of its relative importance in topology and because, as a concept, it bears similarities to what appears to be Piaget's notion of continuity. Openness and closedness of curves were chosen because the topic is found in many elementary school textbooks.

The Test

The measuring instrument is composed of six items (see Appendix). Each item consists of a model figure either drawn on a 5 x 8 notecard or made of one-eighth inch solder glued to a 5 x 8 piece of one-eighth inch thick masonite, and three copies made in the same manner as the models, drawn or made with solder, and

presented on 5 x 8 notecards or masonite, respectively. Copy A is homeomorphic, or topologically equivalent, to the model. Copies B and C, while not strictly Euclideanly equivalent to the model, could represent attempts at Euclidean equivalence. That is, they preserve as many Euclidean properties of the model as is consistent with the fact that they have been altered so as to eliminate a particular topological property which the model possessed. Thus copies B and C preserve properties such as straightness, curvature, line segment length, and angle size, whereas copy A fails to preserve these properties of the model. However, copy B fails to preserve the topological property of being connected. And copy C represents a variation on the topological properties of openness and closedness of the model.

No two of the model figures are topologically equivalent. Thus the model figures represent the widest possible topological variety possible in a six item test. In addition the test includes both symmetric and non-symmetric figures, curvilinear and rectilinear figures, and several variations on openness and closedness.

The Sample

The sample consisted of 90 subjects, 30 from each of the ages 4, 6, and 8. At each age level children were selected randomly, with the age constraint, from three sources. The 6 and 8 year old groups were composed

of children from a rural county school, an Athens, Georgia private academy, and an Athens public school, ten children from each source at each age level. The 4 year old group was composed of children from two nurseries and a day-care center in Athens. All age levels were heterogeneous with regards to sex and race. Mean CA was 58 months for the 4 year olds, 79 months for the 6 year olds, and 106 months for the 8 year olds. Age groups for the study were chosen with consideration given to Piaget's theory of cognitive development to reflect behaviors in successive stages of development.

Administration

During personal interviews the model figures were presented one at a time and in random order to each child. As a model figure was presented to a child for examination, the child was encouraged to remember the model because it would be taken away. After the child had examined the figure to his satisfaction, the model was taken away and the child was confronted with the three copies of the model. The child was asked to select the copy "most like" the model and the "worst" copy of the model.

The test was administered in four modes in the following order to each subject:

(1) tactile examination of model, tactile selection from copies;
(2) visual examination of model, tectile selection from copies;

(3) tactile examination of model, visual selection from copies;
(4) visual examination of model, visual selection from copies.

During the portions of the test the subject wore a blindfold. The test modes were administered in the order indicated above in order to minimize the child's viewing the figures prior to feeling them. Because of the limited attention span of the children, it was necessary to administer the test in two sittings. Modes (1) and (2) above were given during the first session which lasted approximately 25-35 minutes. The remaining two modes, (3) and (4), were given during a second session which usually lasted 15-25 minutes. Generally the two testing sessions were two days apart. In no instance were both sessions held on the same day.

In addition to selecting copies "most like" the model and "worst" copies of the model, each child was asked to draw copies of the last model examined in the tactile examination - tactile selection test and in the visual examination - visual selection test, modes (1) and (4) respectively. Following the selection of the "most like" and the "worst" for all items in mode (1), the last model examined was presented to the child again for tactile re-examination. He was asked to draw a picture of what he thought it looked like based upon his tactile exploration. When the child indicated that he was ready, the mask was raised and the child made

his drawing. If he wished to feel the figure again, he was allowed to stop drawing, replace the mask, and feel the figure once more. He could then raise his mask and complete his drawing. He was not allowed to view the model figure until after his drawing was completed. After the completion of the mode (4) portion of the test, the child was also asked to re-examine the last model he had viewed, in preparation for drawing his own copy of the model. He was allowed to interrupt his drawing once and re-examine the model, but the model was not in view during the actual drawing.

An outline of the administration of mode (4) of the test follows.

> Interviewer: "I am going to show you a picture. Please look at it carefully. Then I am going to show you some drawings which are copies of that picture. You are to tell me which of the drawings is the copy most like the first picture and which drawing is the worst copy of the first picture. There is no right or wrong answer. I just want to know what you think."
>
> Hold up 5 x 8 card with model figure drawn on it (models to be presented in random order).
>
> "Look at this picture. Try to remember what it looks like because I am going to take it away so that you can't see it again."
>
> Hold model in view until child is satisfied. The child may hold the card. Remove model from view. Present the three copies of the model, each on separate 5 x 8 cards. Vary the display of the copies on each item so that copies A, B, and C are not always in the same relative positions.
>
> "Now please tell me which one of these is most like that picture we were just looking at." _____ "Which one do you think is the worst copy?" _____

After all six items have been given and responses recorded, present the last model figure to the child again for visual examination.

"Would you look at this shape again for me? I would like for you to draw a picture of it. en you think you're ready I'll take it away you can draw your picture right here." ent pencil and paper to child. "After start drawing, you may look at the picture one more time if you want to."

Interviews for the other test modes were similar to the one described above, with appropriate adjustments made for the particular mode.

Test Rationale

Since the model figures are not available for examination, either tactile or visual, during the child's selections of the "most like" and the "worst," it is necessary for him to reconstruct an image of the model in his representational space. The reconstructed image is then compared to the copies. The copies themselves represent results of transformations on the model. Copy A is the result of a homeomorphism and hence preserves all topological properties of the model. Copies B and C preserve as many Euclidean properties of the model as is consistent with the fact that they have been altered so as to eliminate specific topological properties which the model possesses.

In a discussion of the drawings of children in what Piaget calls Stage II (about 4;0 - 6;6), he says "the topological relationships ... are now universally applied to all shapes, and in the case of conflict prove stronger

than more recently acquired ones [14, p. 50]." Later in the same discussion he states that although projective and Euclidean relationships begin to develop during Stage II, "representation is still essentially topological [1" p. 50]." Of the Stage I children (about 3;0 - 4;0), reported that "the feature which strikes the children first and foremost in whether a shape is open or closed, and also, ... whether there is anything inside or outside the closed contour [6, p. 61]." Laurendeau and Pirard reported that the child who is unable in a haptic (tactile) perception task to identify a shape he has felt prefers a shape homeomorphic to it [8, p. 62, 64, 67]. Not until Stage III (beginning about 6;6 or 7;0 years), they concluded, do "distinctions made by the child go beyond the elementary level of topological successess to be linked more and more with the Euclidean or metric characteristics ... [8, p. 110]."

The above statements reflect Piaget's hypothesis that the representational space of the child is first topological in nature, later developing to include Euclidean concepts. If topological concepts develop before Euclidean concepts and if invariance under transformation is critical in concept attainment, it seems reasonable to expect children from the four year old age group to designate the copy which is topologically equivalent to the model, that is copy A, as the copy which is "most like" the model. Since both copies B

and C lack specific topological properties of the model, one would expect the same children to select the "worst" copy from copies B and C. If Euclidean concepts begin to gain dominance after about age six, to expect the older children to designate copy B or copy C as best and copy A as worst would seem reasonable.

Design

The study utilized a mixed design with one between- and one within-subjects variable. The between-subjects variable is age and the within subjects variable is test mode. Since specific age groups are under investigation using specific testing modes, both age and testing mode are fixed factors. A diagram of the design is presented in Table 1.

TABLE 1

Design of the Study

	Testing Mode			
	(1)	(2)	(3)	(4)
Age 4	R_{1-30}*	R_{1-30}	R_{1-30}	R_{1-30}
6	R_{31-60}	R_{31-60}	R_{31-60}	R_{31-60}
8	R_{61-90}	R_{61-90}	R_{61-90}	R_{61-90}

*R_{i-j} indicates subjects R_i to R_j.

Data Analysis and Results

The test consisted of six items given to each child in four modes. For each item the child indicated a "most like" preference and a "worst" preference. Thus each child received two scores for each testing mode, a "most like" preference score and a "worst" preference score. Both scores have a range from zero to six.

Means for both types of scores were tested against chance means by using one-tailed "t" tests. The chance mean "most like" preference score is four and the "worst" preference score chance mean is two. "Most like" scores significantly above chance scores would be an indication that the child's representational space is not predominantly topological in nature. "Worst" preference scores significantly above chance would be a somewhat stronger indication of the same thing. For high "worst" preference scores indicate that not only were non-homeomorphic copies of the model considered "most like" the model but also the homeomorphic copy was considered to be the "worst" copy of the model. Actual mean scores and their associated variances are presented in Tables 2 and 3.

For the test modes (3) and (4), mean scores based on the number of times copies which were not topologically equivalent to the model were selected as being "most like" the model were significantly ($p<.01$) above chance means at all age levels. In addition the mean score for the 4 year olds was significantly ($p<.05$) above chance

TABLE 2

Most Like Preference Score Means Compared to Chance Means: Age x Test Mode

	Test Mode							
	(1) Tactile Examination-Tactile Selection		(2) Visual Examination-Tactile Selection		(3) Tactile Examination-Visual Selection		(4) Visual Examination-Visual Selection	
	$\bar{X}$	s^2	$\bar{X}$	s^2	$\bar{X}$	s^2	$\bar{X}$	s^2
Age 4	4.267	1.857	4.406*	1.314	4.844**	1.025	5.422**	.678
Age 6	4.067	1.168	4.267	1.375	5.000**	2.000	4.867**	1.430
Age 8	4.100	1.403	3.867	1.361	5.233**	.599	5.033**	1.206

*significantly ($p<.05$) above chance
**significantly ($p<.01$) above chance

TABLE 3

Worst Preference Score Means Compared to Chance Means: Age x Test Mode

	Test Mode							
	(1) Tactile Examination-Tactile Selection		(2) Visual Examination-Tactile Selection		(3) Tactile Examination-Visual Selection		(4) Visual Examination-Visual Selection	
	$\bar{X}$	s^2	$\bar{X}$	s^2	$\bar{X}$	s^2	$\bar{X}$	s^2
Age 4	2.050	2.040	2.211	1.866	3.522**	2.572	3.711**	1.791
Age 6	2.400	1.300	2.733*	2.961	4.367**	3.482	4.317**	2.39[illegible]
Age 8	2.700*	1.872	2.767**	1.840	4.333**	2.437	3.767**	2.875

*significantly ($p<.05$) above chance
**significantly ($p<.01$) above chance

for mode (2). The remaining five means based on the "most like" criterion were not significantly above chance. The mean scores based on the number of times the topologically equivalent copy was selected as the "worst" copy of the model were significantly ($p<.01$) above chance means for modes (3) and (4) at all age levels and for modes (1) and (2) for 8 year olds. In addition the mean scores for the 6 year olds was significantly above chance ($p<.05$) for modes (1) and (2). The only "worst" preference score means failing to be significantly above chance at either the .05 or .01 level were for the four year olds in modes (1) and (2).

In addition to comparing mean scores to chance means, both "most like" and "worst" preference scores were subjected to analysis of variance. For both types of scores the contrast was age x test mode. The ANOVA information is reported in Tables 4 and 5.

An indicated by Tables 4 and 5, test mode was a significant source of variance for both the "most like" and "worst" preference scores. That is, the test mode had a significant ($p<.01$) effect on the means of these scores. Age had a significant ($p<.05$) effect on the "worst" preference scores but not on the "most like" preference scores. There was no significant interaction between age and mode for either type of score.

Since the F-ratio's indicated significant differences among the mode means for both types of scores and significant

TABLE 4

A x M ANOVA for Most Like Preference Scores

Source of Variation	df	M.S.	F
Between R			
A (age)	2	.448	.209
R/A (residual)	87	2.139	No Test
Within R			
M (mode)	3	21.935	21.935**
AM	6	1.300	1.300
RM/A (residual)	261	1.000	No Test

**.01 level of significance

TABLE 5

A x M ANOVA for Worst Preference Scores

Source of Variation	df	M.S.	F
Between R			
A (age)	2	12.187	3.106*
R/A (residual)	87	3.923	No Test
Within R			
M (mode)	3	70.684	40.623**
AM	6	1.374	.790
RM/A residual	261	1.74	No Test

*.05 level of significance
**.01 level of significance

differences among the age means for the "worst" preference score, Duncan's multiple range test was applied to determine which means caused the differences. Results are displayed in Table 6.

TABLE 6

Differences Indicated By Duncan's Multiple Range Test*

a.

Mode Means: Most Like Preference Scores

(1)	(2)	(3)	(4)
4.145	4.180	5.026	5.007

b.

Mode Means: Worst Preference Scores

(1)	(2)	(3)	(4)
2.383	2.570	4.074	3.932

c.

Age Means: Worst Preference Scores

Age 4	Age 6	Age 8
2.874	3.454	3.392

*Any two means not underscored by the same line are significantly different. Any two means underscored by the same line are not significantly different.

The analysis of variance indicates that test mode effected the children's "most like" and "worst" preference scores. When children chose by using their sense of sight, their respective scores were significantly higher

than their scores where the sense of touch was used for these selections. Also four year old children tend to choose the copy which is topologically equivalent to the model as the "worst" copy less often than the older children.

An important question is "How do these results relate to Piaget's hypothesis that topological representation precedes Euclidean and projective representation?" Since there was no significant interaction between age and mode, the differences attributed to mode are for all ages. Therefore the analysis yields no evidence that age effects the children's "most like" preference scores. Four year olds sacrifice topological properties of the model in their selections as freely as do eight year olds. And as shown in Table 2, mean scores for the four year old age group are at or significantly above chance levels. On the basis of Piaget's hypothesis, one would expect these scores to be significantly below chance levels.

However, the Duncan's Multiple Range Test shows the four year olds more reluctant as a group to select the copy which is topologically equivalent to the model as the "worst" copy. If the representational space of the four year old is predominantly topological, he should be reluctant to designate the topologically equivalent copy of the model as the "worst" copy. But another look at the mean scores in Table 3 shows that the "worst"

preference score means for the four year old group are at or above chance levels in all four testing modes. The means are never below chance levels as would be expected if the four year old's representational space is predominantly topological. Thus despite the fact that the "worst" preference scores for the four year olds tend to be lower than the same scores for the six and eight year olds, they still are too high to support Piaget's theory concerning spatial representation.

Analysis of Drawings

Each child made two drawings. Drawings were first separated according to the sensory mode used for model examination. Then they were separated according to item number. Finally all drawings were evaluated by two independent judges and placed into categories on the basis of properties of the model which the drawing exemplified. These categories are:

Non-Homeomorphic Drawings
(1) No discernible properties,
(2) Some Curvilinearity,
(3) Some Rectilinearity,
(4) Quasi-Projective copy;

Homeomorphic Drawings
(1) Topological properties only,
(2) Some Curvilinearity,
(3) Some Rectilinearity,
(4) Projective copy.

The two major divisions are based upon whether or not a drawing is topologically equivalent to the model which it represents. Within the non-homeomorphic drawings of the model, cases where there was no drawing

attempted or where the attempt resulted in only scribble were placed into the "no discernible properties" class. Cases which were beyond mere scribble and which displayed some elements of curvilinearity or rectilinearity were placed into categories (2) or (3), respectively, of the non-homeomorphic topologies.

It is important to note that the classification of drawings is based upon the properties of the model which the drawing preserves. A drawing showing only curvilinearity while the corresponding model has no curvilinearity would not be placed into category (2) but into category (1). A similar statement applies to rectilinearity. Thus the curvilinearity category is not applicable to the strictly rectilinear models such as model figures (1) and (4) of the test. The rectilinearity category is not applicable to strictly curvilinear models such as model (6). Model figures (2), (3), and (5) contain elements of both curvilinearity and rectilinearity. For these models, the "some curvilinearity" category is subsumed within the "some rectilinearity" category. That is, placement of a drawing into the "some rectilinearity" category may be interpreted that the drawing also demonstrates some curvilinearity. While such an interpretation could have theoretical inconsistencies, it worked quite well in practice for drawings of models used on this test.

The "Topological properties only" category is intended for drawings which are topologically equivalent to the model beyond topological properties. Such a drawing would resemble copy A of the model.

Drawings classified as "projective" copies of their respective models exemplify all topological properties of the model. In addition, straightness, "good" curvature, intersection of lines, convexity, angle presence, and number of angles are preserved. Such non-projective properties as parallelism, angle measure, and length are not necessarily preserved by these drawings. While in a strict mathematical sense some of these drawings are not true projective copies of their respective models, the term 'projective' was considered a good descriptive lable for the category. "Quasi-projective" copies fit the description of projective copies as nearly as is possible and still fail to be topologically equivalent to their respective models. In this respect they resemble copies B and C used in the administration of the test.

Drawings tended to improve as the age of the children increased (see Tables 7 and 8). Drawings by the four year olds based on tactile examination fell into the "non-homeomorphic drawings" category 90 percent of the time. And most of these showed no discernible properties of the models they represent. Only 30 percent of the eight year olds' drawings based on tactile examination fell into the "non-homeomorphic drawings" category. In

TABLE 7

Classification Frequencies for Drawings After Tactile Examination of Model

Properties Preserved	Age 4	Age 6	Age 8
Non-Homeomorphic Drawings			
(1) No discernible properties	17	2	
(2) Some Curvilinearity	3	2	1
(3) Some Rectilinearity	5	4	3
(4) Quasi-Projective Copy	2	5	5
Total; Percent of Age Level	27; 90%	13; 43%	9; 30%
Homeomorphic Drawings			
(1) Topological Properties Only		1	
(2) Some Curvilinearity	2		1
(3) Some Rectilinearity	1	5	3
(4) Projective Copy		11	17
Total; Percent of Age Level	13; 10%	17; 57%	21; 70%

TABLE 8

Classification Frequencies for Drawings After Visual Examination of Model

Properties Preserved	Age 4	Age 6	Age 8
Non-Homeomorphic Drawings			
(1) No Discernible properties	6	2	
(2) Some Curvilinearity	1		
(3) Some Rectilinearity	11	2	
(4) Quasi-Projective Copy		4	3
Total; Percent of Age Level	18; 60%	8; 27%	3; 10%
Homeomorphic Drawings			
(1) Topological Properties Only		1	
(2) Some Curvilinearity	3		2
(3) Some Rectilinearity	7	5	3
(4) Projective Copy	2	16	22
Total; Percent of Age Level	12; 40%	22; 73%	27; 90%

contrast to those of the four year olds, the drawings of the eight year olds displayed an ability to draw projective copies of the models. In light of this ability, it is hypothesized that, given the opportunity, eight year olds could have selected a projective or Euclidean copy as "most like" the model figure.

This hypothesis is supported by the fact that 90 percent of the eight year olds drew homeomorphic, tending to be projective, copies following the visual portion of the test. That they chose non-homeomorphic copies as "most like" the models and homeomorphic copies as "worst" copies of the models more often than chance suggests that even though they possibly were capable of preserving both topological and non-topological properties, they preferred the copies to have the rectilinearity and/or curvilinearity when forced to discard some properties of the model.

Drawings of the four year olds were generally better after the visual portion of the test than after the tactile portion. Although 60 percent of the drawings were still in the non-homeomorphic category, there was improvement in representing some discernible properties of the models. However, drawings as analyzed in this study did not indicate that the representational space of four year olds is predominantly topological in nature.

Additional comparisons of the drawings of individual children with their selections of "most like" and "worst"

copies of the particular models which they drew suggested the following hypothesis. It is not attention to topological properties, per se, which enables children to draw homeomorphic copies. It could be instead the increasing coordination of Euclidean or projective properties which produces the homeomorphic drawings. For proper coordination of projective properties results in automatic preservation of topological properties.

CONCLUSIONS

Attempts to confirm Piaget's theory of the child's conception of space have largely taken the form of replications. There have been few questions raised about the appropriateness of his tasks to measure topological concepts. Replicators such as Lovell [9], Dodwell [3], Laurendeau and Pinard [8], and Peel [13] vary Piaget's tasks only slightly. Consequently the fact that they obtain results similar to Piaget's should not be surprising. Replications are necessary but cannot provide information for making generalizations. They cannot provide information for deciding whether or not the phenomena observed by Piaget are task specific.

This study was an attempt to provide such information. Results do not support the theory that topological concepts develop prior to Euclidean and projective concepts in the child's representational space. None of the mean scores in Tables 2 or 3 indicate that the four year old's

representational space is predominantly topological. In modes (2), (3), and (4) four year olds actually preferred the non-homeomorphic copies of the model as "most like" the model significantly above chance expectation. In addition they selected the homeomorphic copy as the "worst" copy of the model significantly above chance levels in modes (3) and (4). Analysis of Variance and Duncan's Multiple Range Test failed to produce evidence that topological concepts dominate the four year old's representational.

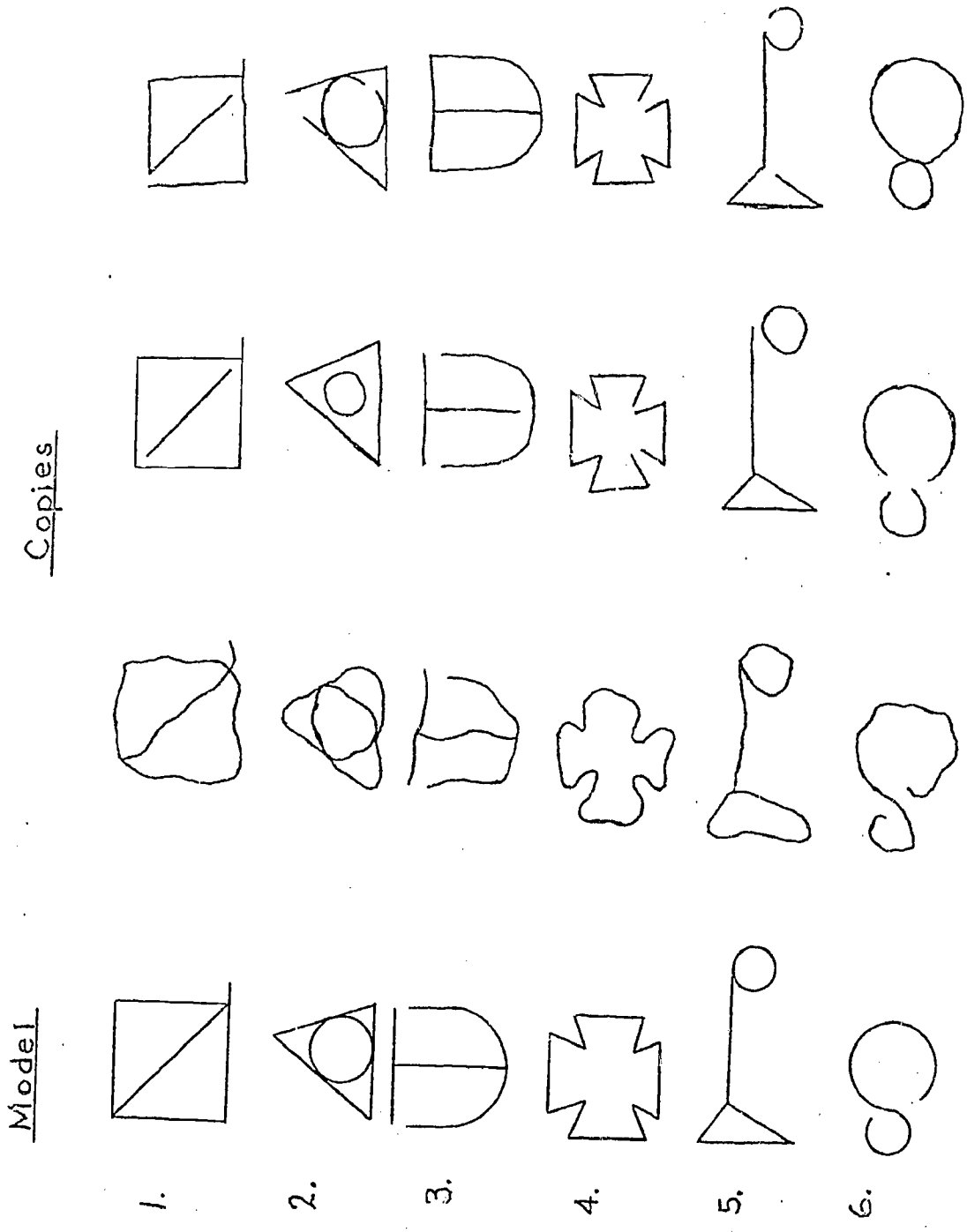
Ninety percent of the drawings made by the four year olds after tactile examination failed to be topologically equivalent to their respective models. The percent of drawings for the four year olds which were topologically equivalent to their respective models increased to 40 percent after visual examination. Yet not one four year old made a drawing which preserved only topological properties. It is hypothesized that it is the increasing coordination of the projective and Euclidean concepts which produce the homeomorphic drawings.

This study should not be interpreted as a refutation of Piaget's theory of the development of representational space. Only two topological properties were under consideration, namely connectedness and openness and closedness of curves. Studies are needed to investigate other topological properties. Also, other tasks should be used to investigate the properties of concern in this study.

Piaget's The Child's Concertion of Space has not received nearly the attention that his The Child's Conception of Number has received. Yet the geometry content in the elementary school is changing rapidly. Piaget's theory that topological properties develop first in the child is increasingly used as a rationale for structuring the geometry sequence. A theory such as Piaget's, with its possible implications for the elementary mathematics curriculum should be dealt with more thoroughly than has been the case and from various viewpoints, mathematical as well as psychological.

APPENDIX

Test Items



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